Krylov Complexity in Free and Interacting Scalar Quantum Field Theories

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- Motivation: **Krylov complexity** is a measure of **operator growth** which has been shown to capture signatures of **chaos** in quantum many-body systems [Parker, et al. (2018)].

- Abstract: In arXiv:2212.14702 (with Viktor Jahnke, Keun-Young Kim and Mitsuhiro Nishida), we study Krylov complexity in free and interacting massive scalar QFTs in $\mathbb{R}^{1,d-1}$ at finite temperature.

(1) Lanczos Coefficients: $a_n\,, b_n$

We are interested in studying the time evolution of Heisenberg operators $\hat{\mathcal{O}}(t)$ in a quantum system with Hamiltonian $\hat{H}$:

$$\hat{\mathcal{O}}(t) = e^{it\hat{H}}\hat{\mathcal{O}}e^{-it\hat{H}} = \hat{\mathcal{O}} + it[\hat{H},\hat{\mathcal{O}}] + \frac{(it)^2}{2!}[\hat{H},[\hat{H},\hat{\mathcal{O}}]] + \cdots$$

$$= \sum_{n=0}^{\infty}\frac{(it)^n}{n!}\mathscr{L}^n\hat{\mathcal{O}} \quad , \quad \text{where} \quad \mathscr{L} := [\hat{H},\cdot\,] \text{ is called the}$$

Liouvillian.

With $\mathscr{L}$, we can define a Gelfand-Naimark-Segal (GNS) Hilbert space after choosing an inner product $(\mathscr{L}^m\hat{\mathcal{O}}\,|\,\mathscr{L}^n\hat{\mathcal{O}})$.

In general, the basis $|\mathscr{L}^n\hat{\mathcal{O}})$ is not orthonormal. However, we can find an orthonormal basis, known as the ***Krylov basis*** $(\hat{\mathcal{O}}_m\,|\,\hat{\mathcal{O}}_n) = \delta_{mn}$ using the Gram-Schmidt procedure:

$$|\hat{\mathcal{O}}_{-1}) := |\mathbf{0}) \quad , \quad |\hat{\mathcal{O}}_0) := |\hat{\mathcal{O}})$$

$$|\hat{\mathcal{O}}_n) = \frac{1}{b_n}\Big((\mathscr{L}-a_{n-1})|\hat{\mathcal{O}}_{n-1}) - b_{n-1}|\hat{\mathcal{O}}_{n-2})\Big) \qquad (n\geq 1)$$

In the Krylov basis, the matrix elements of the Liouvillian take the form

$$(\hat{\mathcal{O}}_m|\mathscr{L}|\hat{\mathcal{O}}_n) = \begin{pmatrix} a_0 & b_1 & 0 & 0 & \cdots \\ b_1 & a_1 & b_2 & 0 & \cdots \\ 0 & b_2 & a_2 & b_3 & \cdots \\ 0 & 0 & b_3 & a_3 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

The coefficients $a_n = (\hat{\mathcal{O}}_n|\mathscr{L}|\hat{\mathcal{O}}_n)$ and $b_n = (\hat{\mathcal{O}}_{n-1}|\mathscr{L}|\hat{\mathcal{O}}_n) = (\hat{\mathcal{O}}_n|\mathscr{L}|\hat{\mathcal{O}}_{n-1})$ are called ***Lanczos coefficients***.

If $\hat{\mathcal{O}}$ is Hermitian, then $a_n = 0$.

(2) Krylov complexity $K_{\hat{\mathcal{O}}}(t)$ and quantum chaos

In the Krylov basis, the time-evolved operator can be written as

$$|\hat{\mathcal{O}}(t)) = \sum_{n=0}^{\infty} i^n\varphi_n(t)\,|\hat{\mathcal{O}}_n) \text{ , where:}$$

$$\varphi_n(t) = \frac{1}{b_n}\Big(ia_{n-1}\varphi_{n-1}(t) - \partial_t\varphi_{n-1}(t) + b_{n-1}\varphi_{n-2}(t)\Big) \qquad (n\geq 1)$$

The ***Krylov complexity*** of the operator $\hat{\mathcal{O}}$ is defined as

$$K_{\hat{\mathcal{O}}}(t) := \sum_{n=0}^{\infty} n\,|\varphi_n(t)|^2$$

Krylov complexity measures the growth or "spread" of the operator $\hat{\mathcal{O}}$ in ***Krylov (sub-)space*** $\mathscr{H}_{\hat{\mathcal{O}}}$, the space spanned by the Krylov basis $\{\,|\hat{\mathcal{O}}_n)\}$.

Chaos in Quantum Many-Body Systems

In [Parker, et al. (2018)] it was conjectured that in the thermodynamic limit, the b_n of generic non-integrable quantum many-body systems with local interactions should grow as fast as possible,

$$b_n \sim \alpha\, n + \gamma \qquad (n\to\infty)$$

In this case, the Krylov complexity should have an exponential growth $K_{\hat{\mathcal{O}}}(t) \sim e^{\lambda_K t}$, with growth-rate λ_K providing a tighter bound than the Maldacena-Shenker-Stanford (MSS) bound on Lyapunov exponents λ_L of OTOCs

$$\lambda_L \leq \lambda_K \leq 2\pi T$$

(3) Krylov complexity $K_{\hat{\mathcal{O}}}(t)$ in QFTs

Q: What is the behavior of b_n and Krylov complexity in QFTs?

To answer this, in arXiv:2212.14702 we focus on the ***Wightman-ordered two-point function*** for a scalar field:

$$\Pi^W(t,\mathbf{x}) := \langle\phi(t-i\beta/2,\mathbf{x})\phi(0,\mathbf{0})\rangle_\beta$$

where $\langle\,\cdot\,\rangle_\beta$ is the thermal expectation value. Its Fourier transform is the ***Wightman power spectrum***

$$f^W(\omega) := \int dt\,\Pi^W(t,\mathbf{0})e^{i\omega t} = \frac{1}{\sinh(\beta\omega/2)}\int\frac{d^{d-1}\mathbf{k}}{(2\pi)^{d-1}}\rho(\omega,\mathbf{k})$$

where $\rho(\omega,\mathbf{k})$ is the spectral density. From the power spectrum, it is possible to compute the ***moments***

$$\mu_{2n} = \frac{1}{2\pi}\int d\omega\,\omega^{2n}f^W(\omega)$$

and from them, the Lanczos coefficients b_n, according to

$$b_1^{2n}\cdots b_n^2 = \det(\mu_{i+j})_{0\leq i,j\leq n}$$

where(μ_{i+j}) is a Hankel matrix of moments.

The ***autocorrelation function*** $C(t) := \varphi_0(t) \equiv \Pi^W(t,\mathbf{0})$ is the starting point for finding $\varphi_n(t)$, from which it is possible to compute the Krylov complexity $K_\phi(t)$.

The spectral density $\rho(\omega,\mathbf{k})$ contains the physical information about the theory. For example, for a free massive scalar

$$\rho(\omega,\mathbf{k}) = \frac{\mathcal{N}}{\epsilon_\mathbf{k}}\Big(\delta(\omega-\epsilon_\mathbf{k}) - \delta(\omega+\epsilon_\mathbf{k})\Big)$$

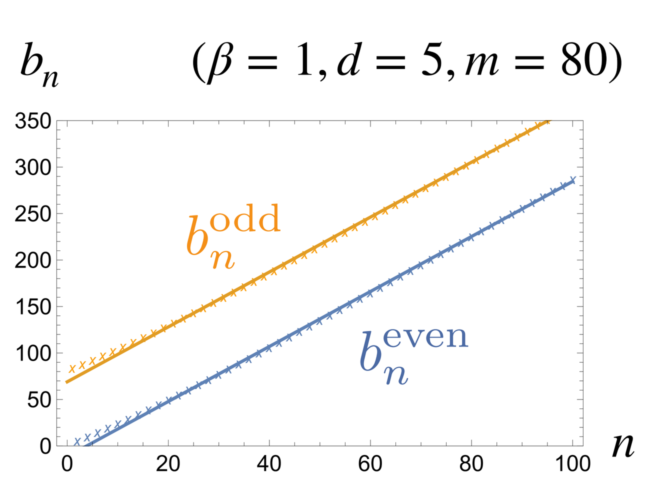
with $\epsilon_\mathbf{k} := \sqrt{|\mathbf{k}|^2+m^2}$.

(4) Free massive scalar QFTs with unbounded spectrum

The power spectrum for a massive scalar field is given by

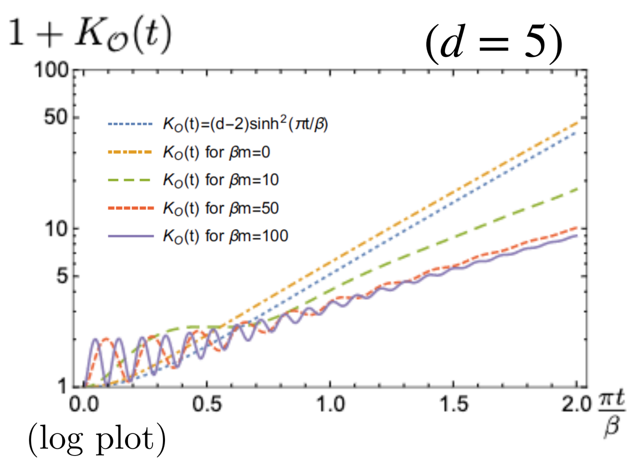
$$f^W(\omega) \sim \Theta(|\omega|-m)(\omega^2-m^2)^{(d-3)/2}/|\sinh(\beta\omega/2)|$$

The **mass** induces a dimerization, or **"staggering"**, of the Lanczos coefficients b_n



$$b_n \sim \begin{cases} \frac{\pi}{\beta}n + \gamma_{\text{odd}} & (n \text{ odd}) \\ \frac{\pi}{\beta}n + \gamma_{\text{even}} & (n \text{ even}) \end{cases} \qquad \text{where:} \quad |\gamma_{\text{odd}} - \gamma_{\text{even}}| \sim m$$

The presence of the mass **breaks the smoothness** of the Lanczos coefficients, and **decreases the growth rate** of the Krylov complexity:



$$K_\phi(t) \sim e^{\tilde{\lambda}_K t} \qquad \left(1.5 \leq \frac{\pi t}{\beta} \leq 2\right)$$

$$\tilde{\lambda}_K \begin{cases} \leq 2\pi/\beta & (m>0) \\ = 2\pi/\beta & (m=0) \end{cases}$$

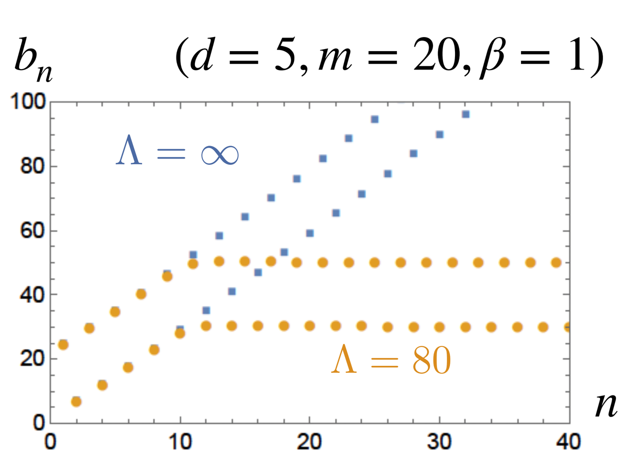
[Dymarsky & Smolkin (2021)]

(5) Free massive scalar QFTs with bounded spectrum

We can introduce a **UV cutoff** in the power spectrum, such that

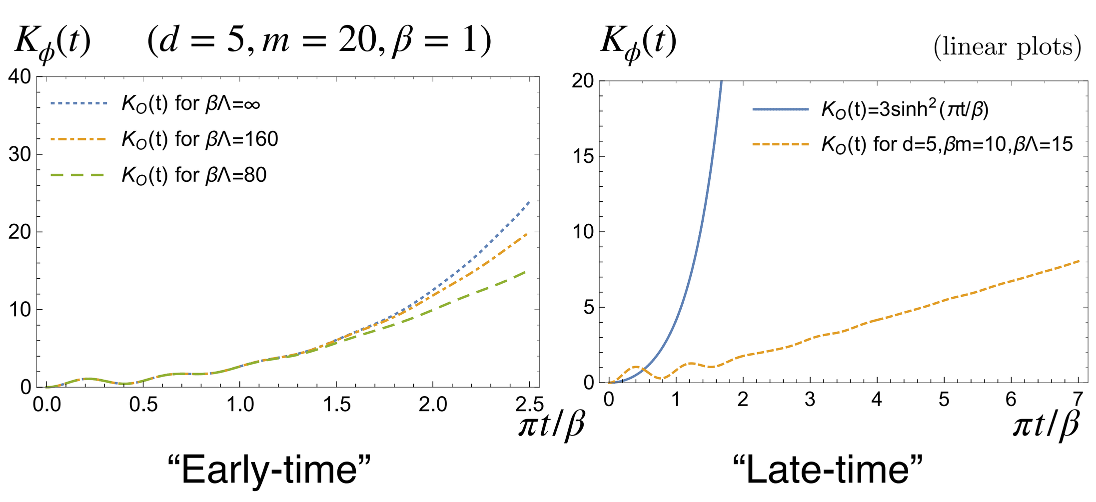
$$f^W(\omega) = 0 \quad (|\omega| > \Lambda)$$

This causes a **saturation** of the Lanczos coefficients



$$b_\pm^{\text{sat}} = \frac{(\Lambda \pm m)}{2} \qquad (n_{\text{sat}} \sim O(\Lambda))$$

consistent with lattice computations ([Avdoskhin, Dymarsky & Smolkin (2022)]) and a **transition** in the Krylov complexity from an exponential to a **linear growth**:



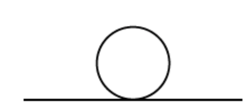
The UV cutoff affects primarily the large-n behavior of the Lanczos coefficients b_n and the late-time behavior of Krylov complexity $K_\phi(t)$.

(6) Interacting scalar in 4d

We consider two types of perturbations in $d=4$:

$$L_{\text{int}} = \frac{g_\ell\,\phi^\ell}{\ell\,!}$$

a) Marginally irrelevant deformation ($\ell = 4$)

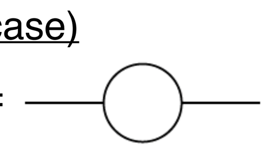
In this case, the one-loop self-energy is $\Pi_E =$ 

This amounts to a change in the **effective mass** of the theory by a thermal mass contribution

$$m_{\text{eff}} = \sqrt{m^2 + m^2_{\text{therm}}} = \sqrt{m^2 + g_4/(24\beta^2)}$$

Thus, we obtain similar results to the free massive case (4).

b) Relevant deformation ($\ell = 3$) (massless case)

In this case, the one-loop self energy is $\Pi_E =$ 

The Lanczos coefficients exhibit a **staggering** that decreases with n